A Column-Generation Approach to Lower Bounds for Resource Leveling Problems

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Outline

Resource leveling problems in project management

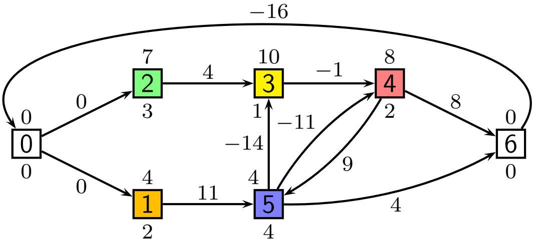
- Project consists of activities $i \in V$ with durations p_i
- Minimum time lags δ_{ij} between start times S_i, S_j of activities i, j
- Project must be completed within deadline $\overline{d}$
- Activities i require r_{ik} units of renewable resources $k \in \mathcal{R}$
- Sought: feasible schedule $S = (S_i)_{i \in V}$ minimizing leveling function

$$f(S) = \sum_{k \in \mathcal{R}} c_k \int_0^{\overline{d}} \varphi(r_k(S,t))\, dt$$

with $r_k(S,t) = \sum_{i \in V : S_i \le t < S_i + p_i} r_{ik}$ and convex function φ

$$(RLP) \begin{cases} \text{Min.} & f(S) & \\ \text{s. t.} & S_j \ge S_i + \delta_{ij} & ((i,j) \in E) \\ & S_i + p_i \le \overline{d} & (i \in V) \\ & S_i \ge 0 & (i \in V) \end{cases}$$

Example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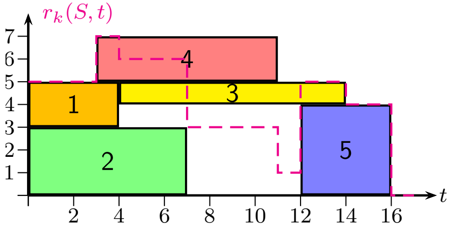
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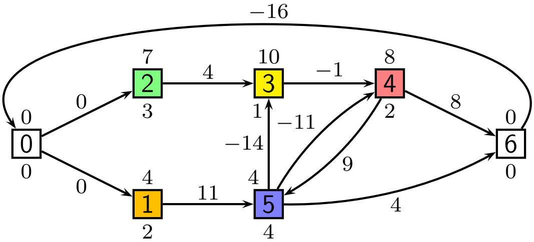
Example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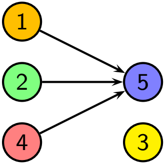
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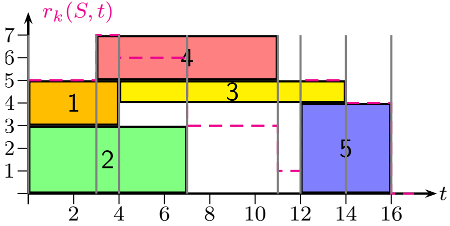
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Reformulation of the problem

- Associate each antichain $A \in \mathcal{A}$ of precedence order $\Theta(D) = \{(i,j) \mid p_i \cdot p_j > 0, d_{ij} \geq p_i\}$ with duration variable x_A
- Encode schedule as a sequence of antichains A with positive durations $x_A > 0$ and resource requirements $r_{Ak} = \sum_{i \in A} r_{ik}$

$$
(RLP') \begin{cases} \text{Min.} & g(x) \\ \text{s. t.} & \sum_{A \in \mathcal{A}: i \in A} x_A = p_i \quad (i \in V) \\ & \sum_{A \in \mathcal{A}} x_A = \overline{d} \\ & x_A \geq 0 \quad (A \in \mathcal{A}) \\ & \text{side constraints} \end{cases}
$$

- Side constraints: feasibility of single-machine problem $1|temp|-$ with set of jobs $J = \{A \in \mathcal{A} \mid x_A > 0\}$

Resource leveling objective functions

General leveling function

$$f(S) = \sum_{k\in\mathcal{R}} c_k \int_0^{\overline{d}} \varphi(r_k(S,t))\,dt \;\rightarrow\; g(x) = \sum_{A\in\mathcal{A}} \underbrace{\left(\sum_{k\in\mathcal{R}} c_k \varphi(r_{Ak})\right)}_{=:c_A} x_A$$

is linear function in duration variables x_A

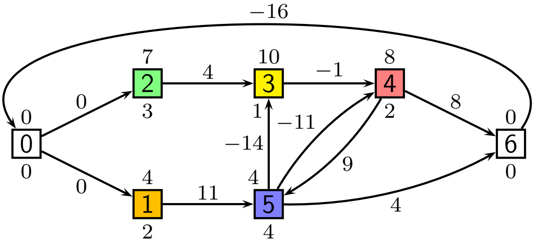
- Total overload cost

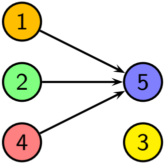
$$\varphi(r_k(S,t)) = [r_k(S,t) - \overline{r}_k]^+ \;\rightarrow\; c_A = \sum_{k\in\mathcal{R}} c_k [r_{Ak} - \overline{r}_k]^+$$

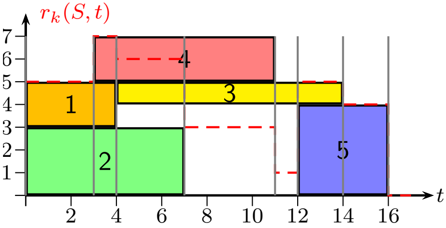
- Total squared utilization cost

$$\varphi(r_k(S,t)) = r_k^2(S,t) \;\rightarrow\; c_A = \sum_{k\in\mathcal{R}} c_k r_{Ak}^2$$

Example revisited for total squared utilization cost







Antichain A	x_A	$r^2_{Ak} \cdot x_A$
$\{1,2\}$	3	$5^2 \cdot 3 = 75$
$\{1,2,4\}$	1	$7^2 \cdot 1 = 49$
$\{2,3,4\}$	3	$6^2 \cdot 3 = 108$
$\{3,4\}$	4	$3^2 \cdot 4 = 36$
$\{3\}$	1	$1^2 \cdot 1 = 1$
$\{3,5\}$	2	$5^2 \cdot 2 = 50$
$\{5\}$	2	$4^2 \cdot 2 = 32$
Σ	$\overline{d} = 16$	$g(x) = 351$

Linear relaxation and column generation principle

- Relaxation of side constraints in (RLP') yields linear program with huge number of decision variables x_A $(A \in \mathcal{A})$

$$(LP)\begin{cases}\text{Min.} & \sum_{A\in\mathcal{A}} = c_A \cdot x_A & & \\ \text{s. t.} & \sum_{A\in\mathcal{A}:i\in A} x_A = p_i & (i \in V) & u_i \\ & \sum_{A\in\mathcal{A}} x_A = \overline{d} & & v \\ & x_A \geq 0 & (A \in \mathcal{A}) & \end{cases}$$

- Solve (LP) by column generation
  - Compute some initial basic solution
  - In each iteration determine nonbasic variable with negative reduced cost by solving an appropriate pricing problem and perform a pivot
  - Terminate procedure when all reduced costs are nonnegative

Reduced costs and optimality condition

- Dual of (LP)

$$(D)\left\{\begin{array}{ll}\text{Max.} & \sum_{i\in V} p_i \cdot u_i + \overline{d} \cdot v \\ \text{s. t.} & \sum_{i\in A} u_i + v \leq c_A \quad (A \in \mathcal{A})\end{array}\right.$$

- Hence reduced costs are

$$\zeta_A = c_A - \sum_{i\in A} u_i - v \quad (A \in \mathcal{A})$$

- Let B be basic matrix to current basic solution x; then simplex multipliers u, v computed as

$$\begin{pmatrix} u \\ v \end{pmatrix} = (B^\top)^{-1} \begin{pmatrix} c^B \\ 0 \end{pmatrix}$$

- Sufficient optimality condition: $\min_{A\in\mathcal{A}} \zeta_A \geq 0$

Pricing problem

- Determine (nonbasic) index A^* with $\zeta_{A^*} = \min_{A \in \mathcal{A}} \zeta_A$
- Introduce binary variable y_i with $y_i = \mathbb{1}_{A^*}(i)$ and define $r_k(y) := \sum_{i \in V} r_{ik} y_i$

Pricing problem: ICP

$$
(PP(u,v)) \begin{cases} \text{Min.} & \zeta_A = \underbrace{\sum_{k \in \mathcal{R}} c_k \cdot \varphi(r_k(y))}_{=c_A} - \sum_{i \in V} u_i y_i - v \\ \text{s. t.} & y_i + y_j \leq 1 \quad ((i,j) \in \Theta(D)) \\ & y_i \in \{0,1\} \quad (i \in V) \end{cases}
$$

- $(PP(u,v))$ represents concave stable set problem on perfect graph (comparability graph of $\Theta(D)$)

Preprocessing

1. Replace positive completion-to-start-time lags $\delta_{ij} - p_i > 0$ by dummy activities with durations $\delta_{ij} - p_i > 0$
2. Identify unavoidable antichains A, which must be in execution in any feasible schedule, i. e., $x_A > 0$ for all feasible x

Proposition

Let $\emptyset \neq A \subseteq V$. Then all activities $i \in A$ are processed in parallel during at least $p(A) = \max\{0, \min_{i,j \in A}(d_{ij} + p_j)\}$ time units. The bound is tight, i. e., there always exists a feasible schedule with $x_A = p(A)$.

- Due to $p(A) = \min_{i \in A} p(A \setminus \{i\})$ the antichains A with $p(A) > 0$ can be computed recursively as cliques of the graph $G = (V, E')$ with edge set $E' = \{\{i, j\} \mid p(\{i, j\}) > 0\}$

Experimental performance analysis

- Testsets j10, j20, j30 with 270 instances each (Kolisch et al. 1999)
- Variation of deadline factor: $DF \in \{1.0, 1.1, 1.5\}$
- Lower bounds compared to optimum values published by Rieck et al. (2012) and Kreter et al. (2014)
- Tested versions of column generation
    - CG1: without preprocessing
    - CG2: completion-to-start dummy activities
    - CG3: identification of unavoidable antichains
    - CG4: combination of CG2 and CG3
- Preprocessing implemented in C#, column generation implemented under GAMS 24.0 invoking Gurobi 5.0 as MIQP-Solver

Numbers of activities after preprocessing	$10-126$
Mean numbers of pivots during column generation	$11-1221$
Mean CPU times in seconds	$4-715$

Experimental performance analysis

Mean relative deviations from optimum objective function values

Total squared utilization cost

	j10			j20			j30	
DF	1.0	1.1	1.5	1.0	1.1	1.5	1.0	1.1
CG1	7.78 %	6.06 %	1.86 %	8.85 %	5.43 %	1.85 %	9.64 %	5.79 %
CG2	4.43 %	5.23 %	1.83 %	5.51 %	4.73 %	1.83 %	6.16 %	4.91 %
CG3	2.66 %	4.42 %	1.01 %	3.38 %	4.20 %	0.79 %	4.75 %	5.15 %
CG4	1.93 %	3.96 %	0.99 %	2.49 %	3.67 %	0.78 %	3.27 %	4.30 %

Total overload cost

	j10			j20			j30	
DF	1.0	1.1	1.5	1.0	1.1	1.5	1.0	1.1
CG4	2.06 %	4.12 %	0.66 %	2.95 %	3.47 %	0.43 %	3.58 %	4.66 %

Summary

- Reformulation of resource leveling problems based on antichain durations
- Relaxing sequencing side constraints yields large-scale linear program
- Linear program solvable via column generation
- Pricing problem represents concave stable set problem on perfect graph
- Relaxation strengthened by preprocessing techniques
- Mean relative deviations $< 5\,\%$ for all scenarios

Future research

- Investigation of the complexity status of the pricing problem
- Branch-and-bound algorithm for resource leveling problems based on antichain formulation

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