Throughput Analysis of Random Storage Systems: Closest Eligible Location Pays

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***Abstract*—We study the performance of storage and retrieval systems executing single-command cycles under the closest eligible location rule. For each arriving storage or retrieval request, this strategy assigns a storage location with minimum cycle time. The performance of the storage and retrieval system is analyzed in terms of different key performance indicators and indices, such as expected cycle time, expected storage times, or storage-time imbalance. Assuming that the arrivals of the storage requests as well as the arrivals of the retrieval requests follow independent Poisson processes, we propose Markov models for the evolution of the spatial inventory distribution in the storage and develop closed-form expressions for the stationary probabilities of a given storage location being selected for a storage or retrieval request. Based on these probabilities, we derive formulas for the key performance indicators and indices and investigate characteristic curves of the expected cycle time and the storage-time imbalance in a numerical experiment. Some of the results obtained were unexpected and disprove popular premises stated in widely-used approaches for throughput analysis. Comparing our results to those of alternative approaches from the literature shows that the latter tend to significantly underestimate the expected maximum system throughput of storage and retrieval systems. The analysis also provides insights into the crucial role that the assignment of storage locations plays in the performance of storage and retrieval systems.**

I. Introduction

In warehouse design, appropriately dimensioning the storage and retrieval (S/R) system presupposes an accurate model of the system throughput under steady-state conditions. The expected maximum system throughput, calculated from the reciprocal expected cycle time, is largely influenced by the storage and retrieval strategy, which defines the way in which storage and retrieval requests are executed during warehouse operation. Given a set of requests to be processed, the strategy partitions the set into operation cycles of the S/R system and allocates appropriate storage locations to each request. The way in which the latter allocation is done will be referred to as the storage assignment strategy. Disregarding the time savings achieved by optimally allotting storage locations to storage and retrieval requests may heavily bias the throughput analysis. Based on continuous-time Markov chains, we derive analytical results for the expected cycle time and further key performance indicators (KPIs) and indices of random storage systems keeping multiple stock keeping units (SKUs). The storage assignment strategy considered in this paper is the closest eligible location (CEL) rule, which for each arriving request selects a storage location allowing for a minimum cycle time. Applied to storage requests exclusively, this strategy is also called the closest open location (COL) rule.

The remainder of this paper is organized as follows. In Sect. II we review the literature of performance evaluation models for random storage systems operated under some cycle-time aware storage assignment strategy. In Sect. III we introduce the Markov models and present the analytical results for different KPIs of S/R systems. Based on numerical experiments, in Sect. IV we investigate characteristic curves for the expected cycle time and the storage-time imbalance and compare our results to alternative approaches from the literature. A short summary and some remarks on future research avenues are given in Sect. V.

II. Literature Review

The estimation of expected cycle times for random storage systems has been studied extensively in the literature. According to the focus of our study, we only include papers in our review which consider storage and retrieval strategies aiming at small cycle times for random storage systems. In their seminal paper [1], Hausman et al. asserted that in case of stable capacity utilization over time, the COL rule can be approximated reasonably well by the pure random storage (PRS) strategy, which allocates storage locations to storage or retrieval requests in a non-optimized, purely random way. Since then, this assumption has been adopted in many modeling approaches for random storage systems, see, e. g., the handbook [2, pp. 659 f.] or the industry standards [3] and [4]. However, recent simulation studies comparing the COL rule with PRS show that the expected cycle times significantly differ when the storage level is below 100 %, see, e. g., [5, pp. 88 ff.].

The literature on analytical performance evaluation of cycle-time aware storage and retrieval strategies is quite sparse. Yamashita et al. [6] examine a unit-load single-SKU random storage system replenished according to an (s, q) reordering policy. After an exponentially distributed lead time, the q reordered units are stored according to the COL rule in the warehouse. The arrivals of retrieval requests follow a Poisson process, and the storage location to be allocated to an entering retrieval request is chosen at random among the currently occupied storage locations. Starting from these assumptions, the authors model the S/R system as a Markov chain and develop an aggregation approach to reduce the state space and the computational effort for calculating the stationary distribution of the stochastic process. Park and Lee [7] consider a unit-load random storage system with Poisson arrivals of the storage requests, which are served under the COL rule. The authors premise i. i. d. storage times with a finite mean W_ℓ for each SKU $\ell = 1, \ldots, L$. Under these assumptions, the S/R system can be modeled as an Erlang loss system with ordered entry, i. e., an $M/G/N/N$ queueing system with each customer taking the first idle server among the ordered servers representing the N ordered storage locations. Note that the mean storage time of a loading unit only depends on SKU ℓ but is independent of the loading unit's storage location n. Moreover, constant mean storage times for SKU ℓ imply that the arrival rate of retrieval requests must be proportional to the current stock of SKU ℓ. The assumptions of the model may, e. g., be satisfied if for each SKU ℓ the arriving retrieval requests form a pure birth process with stock-dependent intensities and the retrieval requests are served according to the PRS strategy.

Besides these analytical models, there also exist heuristic approaches. Fukunari and Malmborg [8] consider Poisson arrivals of the storage requests and a known average storage time of the stored SKUs, which is again independent of the assigned storage location. To model this S/R system as an $M/M/N$ queueing system, they assume a hypothetical load consolidation scenario, in which all SKUs are relocated after each retrieval so that at any time, only the storage locations nearest to the I/O point are occupied. The occupancy probability of each storage location is then approximated based on the stationary probabilities for the number of customers in the queueing system. Due to the hypothetical load consolidation, this model typically only provides rough estimations for real-life storages. Malmborg [9] improves the accuracy of the model by estimating the expected number of free storage locations with a smaller distance to the I/O point than the furthest occupied location. This approach is based on linear regressions on the results of simulation studies.

To the best of our knowledge there is no analytical model available in the open literature that considers the widely-used storage and retrieval strategy minimizing the cycle time for both storage and retrieval requests. We refer to this storage assignment strategy as the CEL rule. The analytical approaches mentioned above only take the COL rule for storage requests into account but ignore the optimization potential of the analogous rule applied to retrieval requests. They assume random storage allocation for the retrieval requests or known mean storage times of the SKUs. Yamashita et al. [6] argue that the CEL rule is not appropriate for practical use because loading units at unfavorable locations may stay in the storage for a very long time. Although the latter observation is correct, this argument does not form an obstacle to the application of the CEL strategy in practice. To avoid obsolescence of inventory, it suffices to periodically retrieve loading units reaching critical storage times. Aside from these occasional FIFO retrievals, the S/R system can then be operated under the CEL rule, which allows for a significant increase in the maximum throughput of the S/R system. In the next section, we introduce the Markov models which form the basis of our throughput analysis.

III. Mathematical Model

We consider a rack storage under random storage strategy serviced by stacker cranes performing single-command cycles. We assume that storage and retrieval requests of different SKUs $\ell = 1, \ldots, L$ are released according to independent Poisson processes with intensities λ_ℓ and μ_ℓ, respectively, and are executed in the sequence of their arrivals. Storage locations are assigned to arriving storage and retrieval requests following the CEL rule. Note that since this rule selects the eligible storage location with minimum sum of the storage and the retrieval cycle times, it is the optimum online storage assignment strategy under our assumptions. If the storage is full or empty, storage and retrieval requests, respectively, are assumed to be lost. We further suppose that all requests refer to single loading units like pallets and that each storage location can hold exactly one loading unit of an arbitrary SKU. For what follows, we establish the convention that the N storage locations $n = 1, \ldots, N$ are numbered in order of nondecreasing cycle times.

In [10] we proposed an aggregation approach to calculate the storage and retrieval access probabilities of each storage location under these assumptions. To sketch the basic idea of this approach, we briefly recapitulate the model for the special case of homogeneous inventory of a single SKU. Let $\{Y(t) \mid t \geq 0\} = \{(Y_1(t), \ldots, Y_N(t)) \mid t \geq 0\}$ be the stochastic process with state space $E = \{0, 1\}^N$ modeling the evolution of the inventory distributed over the storage locations, where $Y_n(t)$ denotes the Bernoulli variable that equals 1 if storage location n is occupied at time t, and 0 otherwise. The feasible state transitions directly follow from the CEL rule. For each arriving request, the first feasible location in the sequence $n = 1, \ldots, N$ of the storage locations is selected, i. e., for a storage request the first free, and for a retrieval request the first occupied location is chosen. The respective transition rates correspond to the arrival rates λ and μ of storage and retrieval requests for the single SKU. Since the size of the state space grows exponentially with increasing number N of storage locations, the stationary distribution of the homogeneous and irreducible continuous-time Markov chain $\{Y(t) \mid t \geq 0\}$ can only be computed for very small instances. Similar to Yamashita et al. [6], we derive truncated and aggregated versions of process $\{Y(t) \mid t \geq 0\}$ to reduce the computational complexity. For fixed $n \in \{0, \ldots, N\}$, we consider the stochastic process $\{Z^{(n)}(t) \mid t \geq 0\}$ with random variables $Z^{(n)}(t)$ counting the number of occupied locations among the first n storage locations. This process corresponds to the throughput process of an $M/M/1/n$ queueing system with arrival rate λ and service rate μ and thus, the formulas for the stationary probabilities $\pi_k^{(n)}$ are known. According to the PASTA property (Poisson arrivals see time averages, see [11, p. 394]), arriving requests see the stationary distribution and we obtain the following formulas for the stationary probabilities $P_S(n)$ and $P_R(n)$ of storage location n being chosen for a storage and a retrieval request, respectively.

$$P_S(n) = \pi_{n-1}^{(n-1)} - \pi_n^{(n)} \tag{1a}$$

$$P_R(n) = \pi_0^{(n-1)} - \pi_0^{(n)} \tag{1b}$$

Storage location n is assigned to a storage request if and only if storage locations 1 to $n-1$ are occupied and the n-th storage location is free. This holds true exactly if the first $n-1$ but not the first n locations are occupied. Analogously, storage location n is selected for a retrieval request if and only if storage locations 1 to $n-1$ are empty and the n-th storage location contains a loading unit.

In [10] we mentioned that these results extend to the case of multiple SKUs $\ell = 1, \dots, L$. For given storage location $n \in \{0, \dots, N\}$, we define the process $\{Z^{(n)}(t) \mid t \geq 0\} = \{(Z_1^{(n)}(t), \dots, Z_L^{(n)}(t)) \mid t \geq 0\}$ with random variables $Z_\ell^{(n)}(t)$ counting the number of storage locations among the first n locations occupied by SKU ℓ at time t, which again is a homogeneous and irreducible continuous-time Markov chain. By $E^{(n)} = \{(j_1, \dots, j_L) \mid \sum_{\ell=1}^L j_\ell \leq n\}$ we denote the state space of the process $\{Z^{(n)}(t) \mid t \geq 0\}$. The following product-form equations provide the stationary probabilities $\pi^{(n)}_{(j_1,\dots,j_L)}$, where $\rho_\ell = \frac{\lambda_\ell}{\mu_\ell}$:

$$\pi^{(n)}_{(j_1,\dots,j_L)} = \pi^{(n)}_{(0,\dots,0)} \cdot \prod_{\ell=1}^{L} \rho_\ell^{j_\ell}, \ (j_1, \dots, j_L) \in E^{(n)} \quad \text{(2a)}$$

$$\sum_{k=0}^{n} \sum_{\substack{j_1,\dots,j_L: \\ j_1+\dots+j_L=k}} \pi^{(n)}_{(0,\dots,0)} \cdot \prod_{\ell=1}^{L} \rho_\ell^{j_\ell} = 1 \quad \text{(2b)}$$

According to (2b), the normalization constants

$$c(n) := \left(\pi^{(n)}_{(0,\dots,0)}\right)^{-1} = \sum_{k=0}^{n} \sum_{\substack{j_1,\dots,j_L: \\ j_1+\dots+j_L=k}} \prod_{\ell=1}^{L} \rho_\ell^{j_\ell} \quad (3)$$

of $\{Z^{(n)}(t) \mid t \geq 0\}$ can be computed recursively using Buzen's convolution algorithm [12] for the normalization constant of a Gordon-Newell network. Applying the same arguments as for homogeneous inventory, the access probability of storage location n for storage requests is

$$P_S(n) = \sum_{\substack{j_1,\dots,j_L: \\ j_1+\dots+j_L=n-1}} \pi^{(n-1)}_{(j_1,\dots,j_L)} - \sum_{\substack{j_1,\dots,j_L: \\ j_1+\dots+j_L=n}} \pi^{(n)}_{(j_1,\dots,j_L)}. \quad (4)$$

Using the normalization constants, (4) can be rewritten as

$$P_S(n) = \frac{c(n-1)}{c(n)} - \frac{c(n-2)}{c(n-1)}. \quad (5)$$

Whereas the storage access probability of storage location n coincides for all types of SKU $\ell = 1, \dots, L$, the retrieval access probability depends on ℓ. The probability of storage location n being chosen for a retrieval request of item type ℓ is given by

$$\begin{aligned} P_R(\ell, n) &= \sum_{\substack{j_1,\dots,j_L: \\ j_\ell=0}} \pi^{(n-1)}_{(j_1,\dots,j_L)} - \sum_{\substack{j_1,\dots,j_L: \\ j_\ell=0}} \pi^{(n)}_{(j_1,\dots,j_L)} \\ &= \rho_\ell \cdot P_S(n). \end{aligned} \quad (6)$$

For the retrieval access probability of storage location n, we obtain

$$\begin{aligned} P_R(n) &= \frac{1}{\sum_{\ell=1}^L \mu_\ell} \sum_{\ell=1}^{L} \mu_\ell \cdot P_R(\ell, n) \\ &= \rho \cdot P_S(n) \end{aligned} \quad (7)$$

with $\rho = \lambda/\mu = \sum_{\ell=1}^L \lambda_\ell / \sum_{\ell=1}^L \mu_\ell$. Recall that storage and retrieval requests are lost if they cannot be served immediately. Since the probabilities $P_S(n)$ and $P_R(n)$ refer to all arriving requests, the service levels of the S/R system for storage and retrieval requests are given by

$$\alpha_S = \sum_{n=1}^{N} P_S(n) = \frac{c(N-1)}{c(N)} \quad (8)$$

$$\alpha_R = \sum_{n=1}^{N} P_R(n) = \rho \cdot \alpha_S. \quad (9)$$

To obtain the expected cycle time, the storage and retrieval access probabilities referring to requests that enter the S/R system have to be considered. According to the service levels in (8) and (9), the expected cycle time for single-command cycles equals

$$\begin{aligned} \bar{t} &= \sum_{n=1}^{N} t(n) \left[\frac{\lambda}{\lambda+\mu} \cdot \frac{P_S(n)}{\alpha_S} + \frac{\mu}{\lambda+\mu} \cdot \frac{P_R(n)}{\alpha_R}\right] \\ &= \sum_{n=1}^{N} t(n) \cdot \frac{P_S(n)}{\alpha_S} \end{aligned} \quad (10)$$

with $t(n)$ denoting the mean of the single-command storage and the single-command retrieval cycle times for storage location n. Note that the normalized access probabilities coincide for storage and retrieval requests, i. e., $P_S(n)/\alpha_S = P_R(n)/\alpha_R$. This finding is plausible because for a specific storage location, storages and retrievals always alternate. The latter observation has been exploited in the analysis of Park and Lee [7].

The mathematical model presented above allows for the evaluation of further indices and KPIs for the steady-state S/R system. First, we consider the expected number of loading units of SKU $\ell = 1, \dots, L$ stored in the warehouse. According to [11, p. 581], this number can be computed as

$$\bar{n}_\ell = \sum_{n=1}^{N} \rho_\ell^n \cdot \frac{c(N-n)}{c(N)}. \quad (11)$$

The mean occupancy rate is given by the ratio of the expected number of occupied storage locations and the capacity N of the storage:

$$\Gamma = \frac{1}{N} \cdot \sum_{\ell=1}^{L} \bar{n}_\ell \quad (12)$$

By applying Little's law, we obtain the expected storage time a loading unit of SKU $\ell = 1, \dots, L$ spends in the S/R system:

$$W_\ell = \frac{\bar{n}_\ell}{\alpha_S \cdot \lambda_\ell} \quad (13)$$

While these indices refer to the S/R system as a whole, we now turn to location-specific parameters. Due to the assumption of unit-load locations and requests, the steady-state occupancy of storage location n is a binary random variable $Y^{(n)}$ and the occupancy rate of storage location n, similarly to [13, p. 89], arises as

$$\begin{aligned}\gamma_n &= P\left(Y^{(n)}=1\right) = E\left[Y^{(n)}\right] \\ &= E\left[\sum_{\ell=1}^{L} Z_\ell^{(n)}\right] - E\left[\sum_{\ell=1}^{L} Z_\ell^{(n-1)}\right] \\ &= 1 - \frac{1}{c(n)}\sum_{\nu=0}^{n-1} c(\nu) + \frac{1}{c(n-1)}\sum_{\nu=0}^{n-2} c(\nu). \qquad (14)\end{aligned}$$

In (14), the occupancy rate γ_n equals the difference of the total expected number of occupied storage locations among the first n and $n-1$ storage locations, respectively. Based on the stationary analysis of the process $\{Z^{(n)}(t) \mid t \geq 0\}$ and some elementary calculus, these expected values can be written in terms of the normalization constants. Once the occupancy rate of each storage location n is known, the expected storage time of a loading unit in storage location n is obtained according to Little's law as follows:

$$w_n = \frac{\gamma_n}{\lambda \cdot P_S(n)} \qquad (15)$$

Finally, we examine the special case of $\rho_\ell = 1$ for all SKUs $\ell = 1, \ldots, L$ in more detail. This case is particularly relevant because for most practical applications, in the long-run the reorder or production rate λ_ℓ should coincide with the demand rate μ_ℓ for each SKU ℓ. Based on our closed queueing network model, after some combinatorial calculus we obtain the following simplified expressions:

$$P_S(n) = P_R(n) = \frac{L}{(n+L-1)(n+L)} \qquad (16a)$$

$$\alpha_S = \alpha_R = \frac{N}{N+L} \qquad (16b)$$

$$\Gamma = \gamma_n = \frac{L}{L+1} \qquad (16c)$$

$$w_n = \frac{(n+L-1)(n+L)}{\lambda(L+1)} \qquad (16d)$$

The first formulas confirm the intuition of access probabilities decreasing in the index n of the storage location. This means that the larger the distance to the I/O point, the less frequently the location is selected for a storage or retrieval request. The storage and the retrieval service levels coincide and deteriorate with growing number L of SKUs. Furthermore, the occupancy rates are identical for all storage locations n and increase in L. Finally, the last equation shows that, as expected, the mean storage time grows with increasing index n of the storage location. However, the storage time remains quadratically bounded in n.

IV. Numerical experiments

In this section, we report on the results of numerical experiments that we performed using our Markov model. First, we present the characteristic curves for the expected cycle time and the storage-time imbalance as functions of the storage capacity N and the number L of SKUs. Then, we compare the outcome of our throughput analysis with the results of alternative approaches from the literature. The analysis is based on the storage rack of a high bay warehouse with 10 levels, 60 bays, square storage locations, and a single I/O point located in the bottom left corner considered in [10]. Assuming that the stacker crane can move simultaneously in horizontal and vertical directions with the same velocity in both directions, the travel times to the storage locations are calculated according to the Tchebychev metric. Furthermore, we suppose that $\lambda_\ell = \mu_\ell$ and hence $\rho_\ell = 1$ for all SKUs $\ell = 1, \ldots, L$.

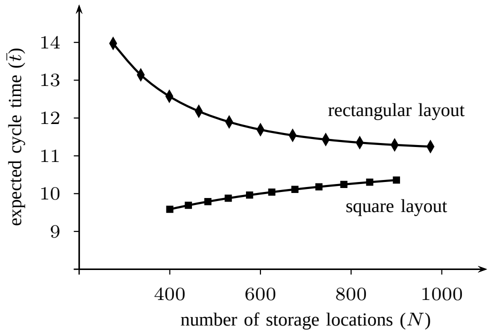

Fig. 1. Expected cycle time as a function of the total number of storage locations

A. Characteristic curves

In what follows, we interpret the KPIs and indices of a storage as functions of the dimensional parameters N and L. For certain KPIs and indices these functions depend on the selected storage and retrieval strategy. In particular, this holds true for the expected cycle time and the storage-time imbalance. We have chosen these two KPIs due to their relevance to storage design and because the CEL rule is intended to minimize the expected cycle time at the cost of an increased storage-time imbalance. In the second subsection we will compare the performance of the CEL rule with respect to these two KPIs against the scenario considered in the model of Park and Lee [7].

At first, we analyze the relationship between the expected cycle time and the storage capacity N. Since the cycle time depends on the arrangement of the storage locations, we consider a square as well as a rectangular storage rack. For the rectangular layout, we start with the storage rack containing 600 storage locations arranged in 10 levels and 60 bays. We then obtain further instances by iteratively adding or removing one level and one bay. In the same way, we construct the instances for the square layout starting with 25 levels and 25 bays. Fig. 1 shows the resulting characteristic curves for both layout types. Somewhat surprisingly, the lines show opposing trends for the two layout types. Whereas for the square storage rack, the expected cycle time grows with increasing capacity, it decreases for the rectangular layout. This effect can be explained as follows. In the square layout, each of the storage locations that is added to an existing storage rack is associated with a maximum cycle time. This observation is not true for the rectangular layout, in which typically the distances between the I/O point and some of the added storage locations are smaller than for certain storage locations contained in the previous configuration.

The characteristic curve in Fig. 2 displays the relationship between the expected cycle time and the number L of SKUs for the original rectangular layout with 600 storage locations. The results show that the expected cycle time is an increasing

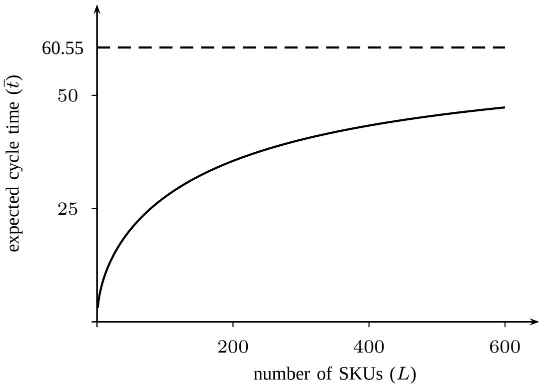

Fig. 2. Expected cycle time as a function of the number of SKUs

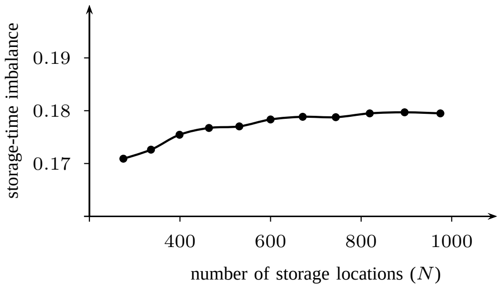

Fig. 3. Storage-time imbalance as a function of the total number of storage locations

and concave function in the number of SKUs stored in the warehouse. In [10] it is mentioned for $\rho_\ell = 1$ for all SKUs ℓ that $\bar{t}$ attains the expected cycle time of the PRS rule as L tends to infinity, the latter cycle time being independent of L.

Next, we turn to the expected storage time of each storage location n. As an aggregate KPI, we propose the storage-time imbalance, which we define to be the percentage of storage locations for which the expected storage time of SKUs exceeds the double of the total mean obtained by averaging over all storage locations. Storage capacity N is varied by considering the same rectangular layouts as before. Fig. 3 reveals that the storage-time imbalance is only little affected by the storage capacity. The curve shows a global trend towards increasing imbalance for larger storage racks. However, note that the function is not consistently increasing since the storage-time imbalance decreases with growing N until the storage time of a further storage location exceeds the threshold.

Finally, we investigate the relationship between the storage-time imbalance and the number L of SKUs for the scenario with 600 storage locations. The characteristic curve in Fig. 4 shows that this relation follows a decreasing, almost linear step function. The more types of SKUs are stored in the warehouse, the less storage locations contain loading units over a long time. Since for $L \to \infty$, the storage behaves like under the PRS rule, the storage-time imbalance tends to zero when the number of SKUs becomes large. In sum, the curves of Figs. 3 and 4 suggest that obsolescence of inventory does not pose a serious obstacle to the application of the CEL rule.

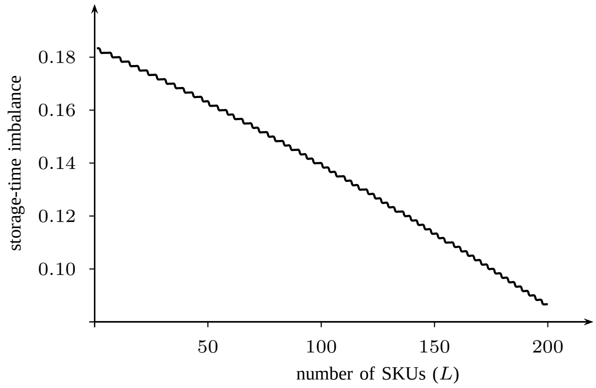

Fig. 4. Storage-time imbalance as a function of the number of SKUs

TABLE I. COMPARISON TO MODELS ASSUMING UNIFORM DISTRIBUTION

Model	MM(1)	MM(3)	MM(5)	MM(10)	MM(15)	UD
$\bar{t}$	3.12	5.34	6.86	9.61	11.69	60.55
δ_{tv}	0.92	0.87	0.83	0.77	0.73	–

B. Comparison with previous approaches

We proceed by putting the performance of the CEL rule into perspective with the results obtained for strategies where locations are, at least to some extent, chosen at random. At first, we revisit the experiments discussed in [10]. In this study, we compared Hausman et al.'s classical assumption of uniformly distributed access probabilities (UD model), i. e., an S/R system operated under the PRS strategy, with the access probabilities arising from our Markov model for the CEL rule (model MM(L), with L as the number of SKUs). Table I displays the expected cycle time $\bar{t}$ for the different models. Moreover, as a measure of inaccuracy for the UD model, we provide the total variation distances δ_{tv} between the uniform distribution and the normalized storage (and retrieval) access probabilities $P_S(n)/\alpha_S$. The results show that when the S/R system is operated under the CEL rule and the number of SKUs is relatively small, assuming a uniform distribution heavily underestimates the maximum system throughput, which seems to be contrary to the assertion of Hausman et al. mentioned in Sect. II. As a consequence, applying the industry standards [3] and [4] may lead to a largely oversized S/R system. We notice, however, that the expected cycle time significantly increases and hence the error of the UD model diminishes when the number L of SKUs increases, as visualized in Fig. 2. This observation is confirmed by the decrease of the total variation distance with growing number L of SKUs. Recall that the normalized storage (and retrieval) access probabilities converge to the uniform distribution as L tends to infinity. However, the convergence may be rather slow. In our example with 600 storage locations, an acceptable total variation distance of 5 % is only attained with more than 2700 types of items. The lessons learnt from the experiments can also be put differently. Given that the UD assumption is satisfied if the storage is operated under the PRS strategy, the large gaps between the expected cycle times provide evidence for the considerable optimization potential of the CEL rule.

Finally, we compare our model to the approach by Park and Lee [7], which represents an S/R system that is run applying

TABLE II. EXPECTED CYCLE TIMES OF PARK AND LEE'S MODEL

Model	PL(1)	PL(3)	PL(5)	PL(10)	PL(15)
$\bar{t}$	31.72	46.56	51.69	56.50	58.07

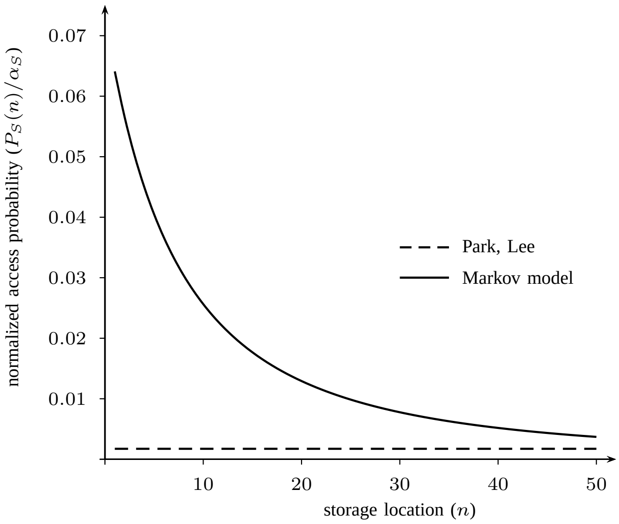

Fig. 5. Comparison of access probabilities

the COL rule to the storage requests. For the purpose of comparison, we calculate the mean storage time W_ℓ yielded by (13) of our model for each SKU ℓ. Subsequently, we insert the values of W_ℓ as parameters into the model of Park and Lee and compute the storage and retrieval access probabilities according to their formulas. The resulting expected cycle times $\bar{t}$ are displayed in Table II, where PL(L) stands for the model of Park and Lee with L SKUs.

The results show that the model by Park and Lee does not underestimate the maximum system throughput as much as the UD model, but the resulting expected cycle times are still significantly higher than those of our Markov model. For the instance with $L = 15$, a closer look to the normalized storage access probabilities $P_S(n)/\alpha_S$ of the first 50 storage locations in Fig. 5 reveals that in our model, the chances of favorable storage locations being assigned to a storage request are significantly higher than those of the Park and Lee model. It came as a surprise to us that the probabilities computed by the approach of Park and Lee only slightly deviate from the uniform distribution. This seems to indicate that their assumptions, especially with respect to i. i. d. storage times, do not comply with a storage and retrieval strategy minimizing the expected cycle time. As it is shown by (16d), the expected storage time w_n of a loading unit is a quadratic function in the location index n, which clearly contradicts the premise of identically distributed storage times.

V. Conclusion and Future Work

In this paper, we proposed Markov models for the system throughput analysis of S/R systems executing single-command cycles under the CEL rule. We showed how the expected cycle time and further KPIs and indices of S/R systems can be calculated efficiently based on aggregate Markov models. As a major finding we could demonstrate that in contrast to a conjecture stated in the literature, inventory obsolescence is not a big issue when applying the CEL strategy. Comparing our results with alternative approaches from the literature reveals that the latter may significantly underestimate the maximum system throughput. Future research will be concerned with extending the models to dual-command cycles and considering batch arrivals of the storage and retrieval requests.

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