Robust Project Scheduling: An Order-Based Approach

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Outline

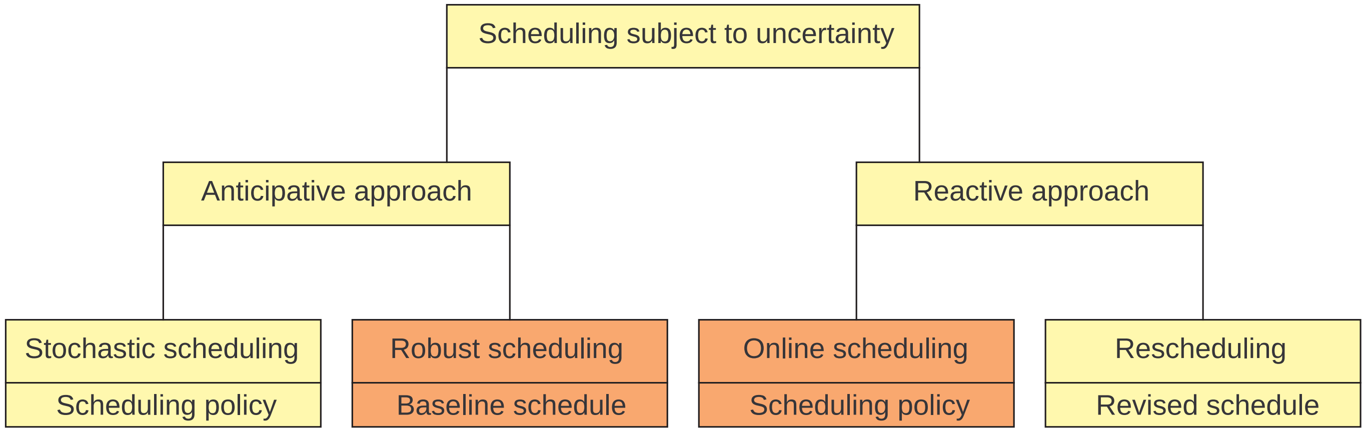
1. Uncertainty in Project Scheduling
2. Robustness and Feasible Relations
3. Solution Method
   3.1 Structural Issues
   3.2 Branch-and-Bound
4. Conclusions

1 Uncertainty in Project Scheduling

Reasons for uncertainty in project scheduling

- Imprecise time or resource estimations (project is a *unique* undertaking)
- Unforeseen downtimes of resources or staff time offs
- Late delivery of raw martials or bought-in parts
- Reworking time

Approaches to cope with uncertainty



Related literature

Scheduling subject to uncertainty:

- Aytug, H., Lawley, M. A., McKay, K., Mohan, S., Uzsoy, R. (2005), Executing production schedules in the face of uncertainties: A review and some future directions. *European Journal of Operational Research 161*, 86–110
- Herroelen, W. S., Leus, R. (2005), Project scheduling under uncertainty: Survey and research potentials. *European Journal of Operational Research 165*, 289–306

Robust project scheduling:

- Herroelen, W. S., Leus, R. (2004), Robust and reactive project scheduling: A review and classification of procedures. *International Journal of Production Research 42*, 1599–1620
- Leus, R. and Herroelen, W. S. (2004), Stability and resource allocation in project scheduling. *IIE Transactions 36*, 667–682
- S. (2005), *Resource Allocation in Project Management.* Springer, Berlin

Algorithmic graph theory:

- Kaerkes, R., Leipholz, B. (1977), Generalized network functions in flow networks. *Operations Research Verfahren 27*, 225–273
- Möhring, R. H. (1985), Algorithmic aspects of comparability graphs and interval graphs. In: Rival, I. (ed.) *Graphs and Orders.* D. Reidel Publishing Company, Dordrecht, pp. 41–101

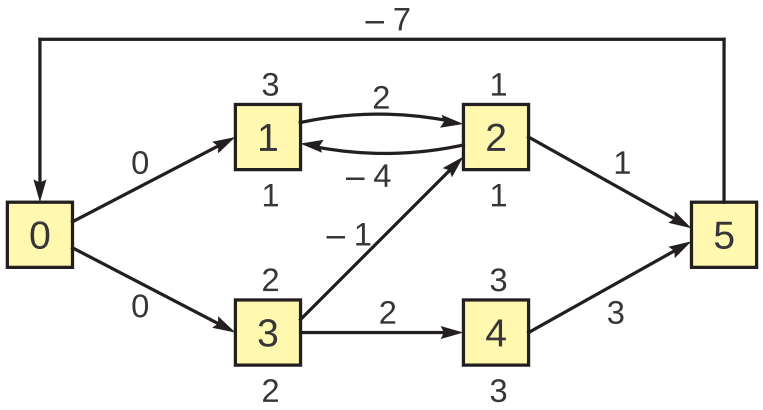
2 Robustness and Feasible Relations

Notations

V	Set of activities $i = 0, 1, \ldots, n, n+1$ of durations p_i
$E \subseteq V \times V$	Temporal relation
δ_{ij}	Minimum time lag between activities i and j
$N = (V, E, \delta)$	MPM network with node set V, arc set E, and arc weights δ_{ij}
$\mathcal{R}$	Set of renewable resources
R_k	Capacity of resource k
r_{ik}	Requirement of activity i for resource k
S_i, $S = (S_i)_{i \in V}$	Start time of activity i, schedule
$r_k(S, t)$	Requirements for resource k at time t given schedule S

Temporal constraints

$$S \in \mathcal{S}_T : \begin{cases} S_j - S_i \geq \delta_{ij} & ((i,j) \in E) \\ S_0 = 0 \end{cases}$$

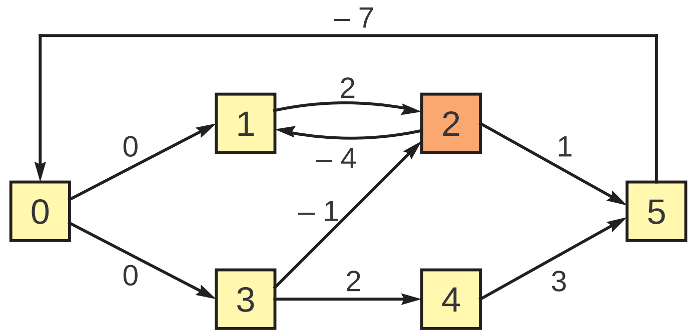


Resource constraints

$$S \in \mathcal{S}_R : \; r_k(S,t) \leq R_k \quad (k \in \mathcal{R}; \; t \geq 0)$$

$$\mathcal{S} := \mathcal{S}_T \cap \mathcal{S}_R$$

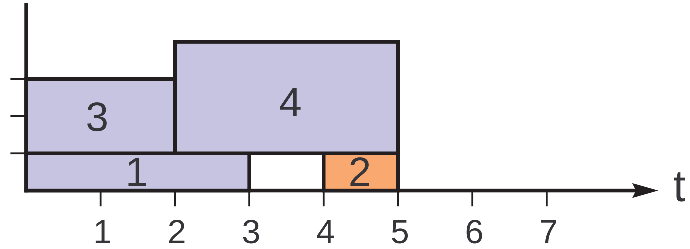
Measure of robustness



Project early and late free floats

$$EFF_i = \min_{(i,j)\in E}(ES_j - \delta_{ij}) - ES_i$$

$$LFF_i = LS_i - \max_{(j,i)\in E}(LS_j + \delta_{ji})$$



Schedule early and late free floats

$$EFF_i(S) = \min_{(i,j)\in E}(S_j - \delta_{ij}) - S_i$$

$$LFF_i(S) = S_i - \max_{(j,i)\in E}(S_j + \delta_{ji})$$

$$f(S) = \sum_{i\in V} w_i^f [EFF_i(S) + LFF_i(S)] = \sum_{i\in V} w_i^f [\min_{(i,j)\in E}(S_j - \delta_{ij}) - \max_{(j,i)\in E}(S_j + \delta_{ji})]$$

- Total weighted free float $f(S)$ measure of schedule robustness

Including resource constraints: Feasible relations

- Break up minimum forbidden sets by precedence relationships (i, j) with $S_j - S_i \geq p_i$
- Precedence relationships form asymmetric (precedence) relation ρ
- Relation polytope $\mathcal{S}_T(\rho) := \{S \in \mathcal{S}_T \mid S_j - S_i \geq p_i \text{ for all } (i, j) \in \rho\}$
- Relation network $N(\rho) = (V, E \cup \rho, \delta^\rho)$ with $\delta^\rho_{ij} = p_i$ for $(i, j) \in \rho$
  $D(\rho)$: Distance matrix of relation network $N(\rho)$
- Precedence relation ρ time-feasible: $\mathcal{S}_T(\rho) \neq \emptyset$
- Time-feasible precedence relation ρ feasible: $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$
- Induced strict order $\Theta(D(\rho)) := \{(i, j) \in V \times V \mid i \neq j,\ d^\rho_{ij} \geq p_i\}$

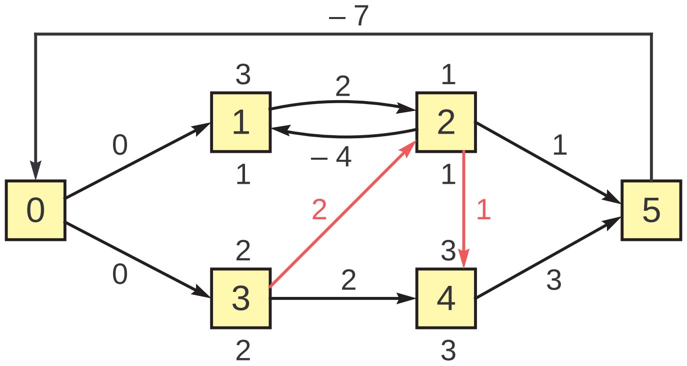
Theorem. Precedence relation ρ is feasible if and only if

(i) relation network $N(\rho)$ does not contain cycle of positive length

(ii) no antichain in strict order $\Theta(D(\rho))$ is forbidden

Example: Precedence relation $\rho = \{(2,4),(3,2)\}$

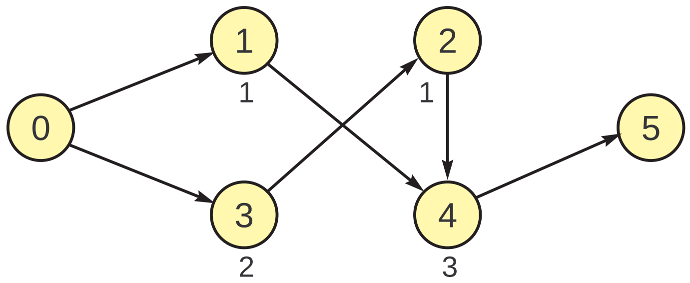
Relation network $N(\rho)$:



Distance matrix $D(\rho)$:

$$D(\rho) = \begin{pmatrix} 0 & 0 & 2 & 0 & 3 & 5 \\ -1 & 0 & 2 & -1 & 3 & 6 \\ -3 & -3 & 0 & -3 & 1 & 4 \\ -1 & -1 & 2 & 0 & 3 & 6 \\ -4 & -4 & -2 & -4 & 0 & 3 \\ -7 & -7 & -5 & -7 & -4 & 0 \end{pmatrix}$$

- Induced strict order $\Theta(D(\rho))$: Transitively reduced precedence graph



- $\subseteq$-maximal antichains: $\{0\}$, $\{1,2\}$, $\{1,3\}$, $\{4\}$, $\{5\}$

Optimization problem

- Precedence relations $(i,j) \in \rho$ influence total weighted free float

$$f(\rho, S) := \sum_{i \in V} w_i^f \big(\min_{(i,j) \in E \cup \rho} [S_j - \delta_{ij}^\rho] - \max_{(j,i) \in E \cup \rho} [S_j + \delta_{ji}^\rho] \big)$$

- **Problem**: Determine feasible relation ρ and (feasible) baseline schedule $S \in \mathcal{S}_T(\rho)$ with maximum total weighted free float $f(\rho, S)$

$$\text{(P)} \quad \left\{ \begin{array}{ll} \text{Maximize} & f(\rho, S) \\ \text{subject to} & \mathcal{S}_T(\rho) \subseteq \mathcal{S} \\ & S \in \mathcal{S}_T(\rho) \end{array} \right.$$

- **Subproblems**:
  - ▷ Temporal scheduling problem for given precedence relation ρ:

    $$\text{Maximize } \{ f(S, \rho) \mid S \in \mathcal{S}_T(\rho) \}$$

  - ▷ Feasibility problem for given precedence relation ρ with $\mathcal{S}_T(\rho) \neq \emptyset$

    $$\mathcal{S}_T(\rho) \overset{?}{\subseteq} \mathcal{S}$$

3 Solution method

3.1 Structural Issues

Properties of objective function and feasible region

- Function $f(\rho, \cdot)$ piecewise linear and concave in S
- Temporal scheduling problem can be transformed into linear program

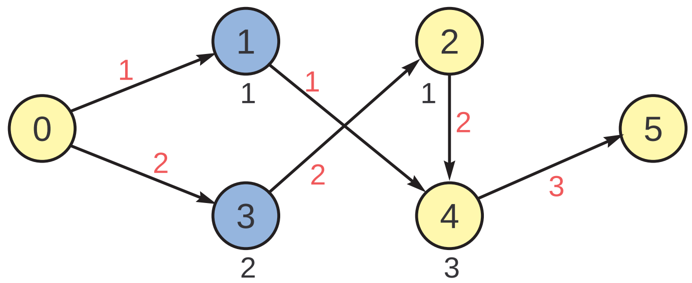
$$
(\mathrm{LP}(\rho)) \quad \left\{ \begin{array}{lll} \text{Maximize} & \sum_{i \in V} w_i^f (x_i^e + x_i^l) & \\ \text{subject to} & S_j - x_i^e \geq \delta_{ij}^\rho & (i \in V,\ (i,j) \in E \cup \rho) \\ & S_j + x_i^l \leq -\delta_{ji}^\rho & (i \in V,\ (j,i) \in E \cup \rho) \\ & S_j - S_i \geq \delta_{ij}^\rho & ((i,j) \in E \cup \rho) \\ & S_0 = 0 & \end{array} \right.
$$

- Function $f(\cdot, S)$ nonincreasing in ρ
- $\mathcal{S}_T(\rho)$ nonincreasing in ρ
- If (P) solvable, there exists optimal and $\subseteq$-minimal feasible relation ρ

Properties of time-feasible relation ρ and strict order $\theta = \Theta(D(\rho))$

- ρ feasible iff weight $w(U_k)$ of maximum-weight antichain U_k in θ with weights r_{ik} no greater than capacity R_k for all $k \in \mathcal{R}$
- Antichain in θ corresponds to stable set in (transitive) precedence graph $G(\theta)$
- Weight of maximum-weight stable set can be determined by computing value of minimum $(0, n+1)$-flow in (transitively reduced) precedence graph $G(\theta)$ with lower node capacities r_{ik}
- Maximum-weight antichain U_k coincides with maximum $(0, n+1)$-node-cut in $G(\theta)$

Example:
Minimum $(0,5)$-flow and maximum $(0,5)$-node-cut for $\rho = \{(2,4),(3,2)\}$



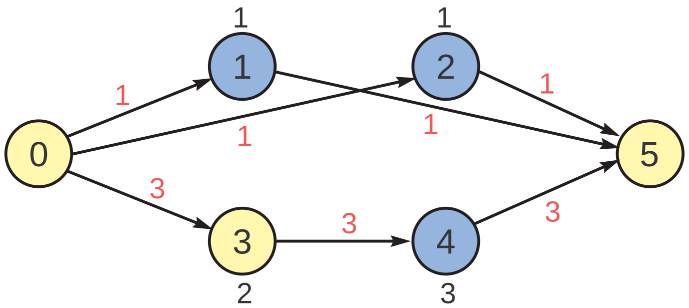
3.2 Branch-and-Bound

Enumeration scheme

initialize list of time-feasible relations $Q := \{\emptyset\}$ and lower bound $f^* := -\infty$;
repeat
  delete some relation ρ from list Q;
  determine maximum-float schedule S by solving temporal scheduling problem LP(ρ);
  if $\Theta(D(\rho')) \not\subseteq \Theta(D(\rho))$ for all $\rho' \in Q$ **and** $f(\rho, S) > f^*$ **then**
    for all $k \in \mathcal{R}$ **do** determine maximum-weight antichain U_k in $\Theta(D(\rho))$;
    if $w(U_k) \leq R_k$ for all $k \in \mathcal{R}$ **then** put $\rho^* := \rho$, $S^* := S$, $f^* := f(\rho^*, S^*)$;
    else
      select some $k \in \mathcal{R}$ with $w(U_k) > R_k$;
      compute set $\mathcal{B}$ of all minimal delaying alternatives for U_k;
      for all $B \in \mathcal{B}$ **do**
        for all $i \in U_k \setminus B$ **do**
          set $\rho' := \rho \cup (\{i\} \times B)$;
          if $\mathcal{S}_T(\rho') \neq \emptyset$ **then** add ρ' on list Q;
until $Q = \emptyset$;
if $f^* > -\infty$ **then return** (ρ^*, S^*);

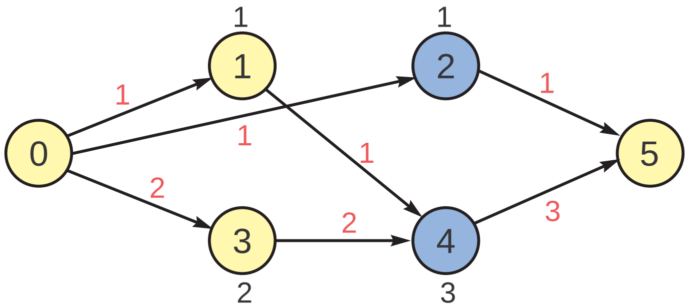
Example: Resource capacity $R = 3$

Iteration 1, root node 0: $\rho = \emptyset$



- $S = (0, 0, 4, 0, 2, 5)$, $f(\rho, S) = 8$
- $U = \{1, 2, 4\}$, $w(U) = 5 > R$
- $\mathcal{B} = \{\{1, 2\}, \{4\}\}$
- $\rho'_1 = \{(1, 4)\}$, $\rho'_2 = \{(2, 4)\}$,
  $\rho'_3 = \{(4, 1), (4, 2)\}$

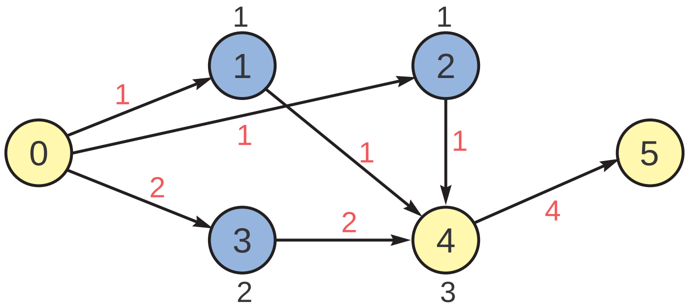
Iteration 2, node 1: $\rho = \{(1, 4)\}$



- $S = (0, 0, 3, 0, 4, 7)$, $f(\rho, S) = 6$
- $U = \{2, 4\}$, $w(U) = 4 > R$
- $\mathcal{B} = \{\{2\}, \{4\}\}$
- $\rho'_{1,1} = \{(1, 4), (2, 4)\}$, $\rho'_{1,2} = \{(1, 4), (4, 2)\}$

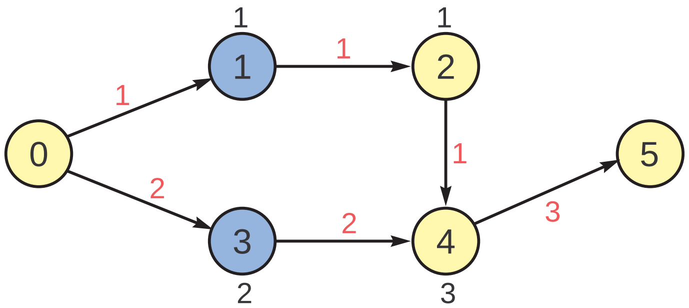
Iteration 3, node 1.1: $\rho = \{(1,4),(2,4)\} \supseteq \rho' = \rho_2 = \{(2,4)\}$

Iteration 4, node 2: $\rho = \{(2,4)\}$



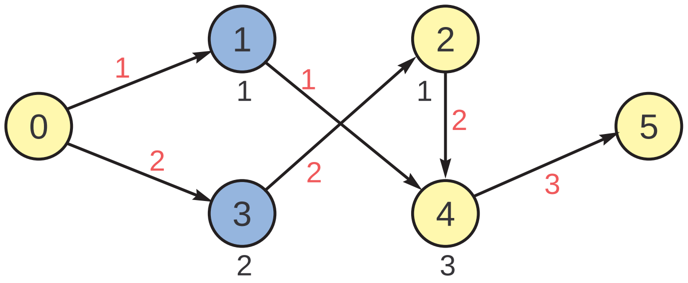
- $S = (0,0,2,0,4,7)$, $f(\rho, S) = 4$
- $U = \{1,2,3\}$, $w(U) = 4 > R$
- $\mathcal{B} = \{\{1\},\{2\},\{3\}\}$
- $\rho'_{2,1} = \{(2,4),(1,2)\}$, $\rho'_{2,2} = \{(2,4),(3,2)\}$,
  $\rho'_{2,3} = \{(2,4),(1,3)\}$, $\rho'_{2,4} = \{(2,4),(2,1)\}$,
  $\rho'_{2,5} = \{(2,4),(2,3)\}$, $\rho'_{2,6} = \{(2,4),(3,1)\}$

Iteration 5, node 2.1: $\rho = \{(2,4),(1,2)\}$



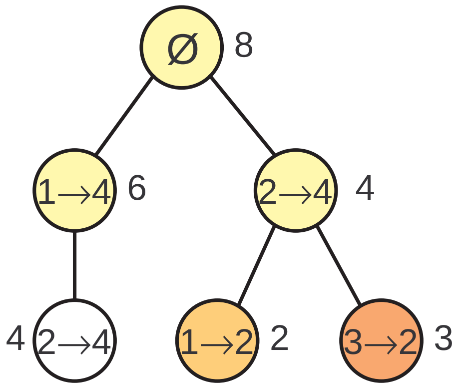
- $S = (0,0,3,0,4,7)$, $f(\rho, S) = 2$
- $U = \{1,3\}$, $w(U) = 3 = R$
- $\rho^* := \rho$, $S^* := S$, $f^* := 2$

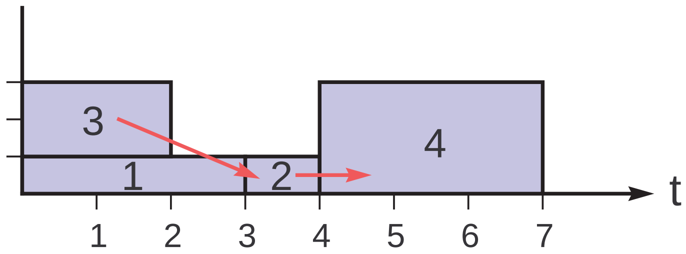
Iteration 6, node 2.2: $\rho = \{(2,4),(3,2)\}$



- $S = (0,0,3,0,4,7)$, $f(\rho, S) = 3$
- $U = \{1,3\}$, $w(U) = 3 = R$
- $\rho^* := \rho$, $S^* := S$, $f^* := 3$

Iteration 7: $Q = \emptyset$, **return** (ρ^*, S^*)





$EFF_1(S^*) = 1$, $EFF_3(S^*) = 1$,
$LFF_2(S^*) = 1$

4 Conclusions

Summary

- Robust project scheduling: robust baseline schedule combined with scheduling policy for resolving resource conflicts during implementation
- Measure of robustness: total weighted free float of schedule
- Temporal scheduling problem: linear program
- Feasibility problem for relation: minimum-flow problem

Further research

- Implementation and testing
- Expansion to sequence-dependent changeover times and cumulative resources
- Application to process scheduling in the chemical industry
- Comparison to alternative approaches under different uncertainty scenarios (rescheduling, pure online scheduling)