Project Scheduling Under Generalized Precedence Relations

A Survey of Structural Issues, Solution Approaches, and Applications

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Outline

Project scheduling problem

- Consider project with n activities $i \in V$ of durations $p_i \in \mathbb{Z}_{\geq 0}$
- Project scheduling problem: assign execution times to each activity i

$$y_i : \mathbb{R}_{\geq 0} \to \{0,1\} \text{ such that } \int_0^\infty y_i(t)\,dt = p_i$$

- Non-preemptive problem: activities cannot be interrupted
  - $\rightarrow$ Solution to scheduling problem specified by start times S_i or completion times $C_i = S_i + p_i$ of all activities $i \in V$
- Preemptive scheduling problem: activities can be interrupted and resumed later on
  - $\rightarrow$ Solution to scheduling problem specified by trajectories

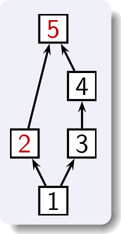
$$z_i : \mathbb{R}_{\geq 0} \to [0,1], \quad t \mapsto z_i(t) = \tfrac{1}{p_i} \int_0^t y_i(t)\,dt$$

for all activities $i \in V$

Precedence relations and resource constraints

- Activities have to be scheduled subject to precedence relations and resource constraints so as to optimize one or several measures of project performance
- Precedence relations: elements (i, j) of binary relation $E \subseteq V \times V$ on activity set V defining conditions on execution times of activities i and j
- Pairs (i, j) may be associated by some extra data like time lags δ_{ij} or execution percentages ξ_i and ξ_j
- Resource constraints: limited availability of manpower, machinery, materials, money, energy supply, ...
- Scheduling goals formulated as objective function(s) in decision variables S_i, C_i or functions y_i, z_i

Types of precedence relations



1. Ordinary precedence relations (Kelley 1961)

   $$(i,j) : S_j \geq C_i$$

2. Generalized precedence relations (Roy 1964)

   $$(i,j,\delta_{ij}) : S_j \geq S_i + \delta_{ij}$$

3. Feeding precedence relations (Kis 2005, Alfieri et al. 2011)

   $$(i,j,\xi_i) : S_j \geq \min\{t \mid z_i(t) = \xi_i\}$$

4. Generalized work precedence relations (Quintanilla et al. 2012)

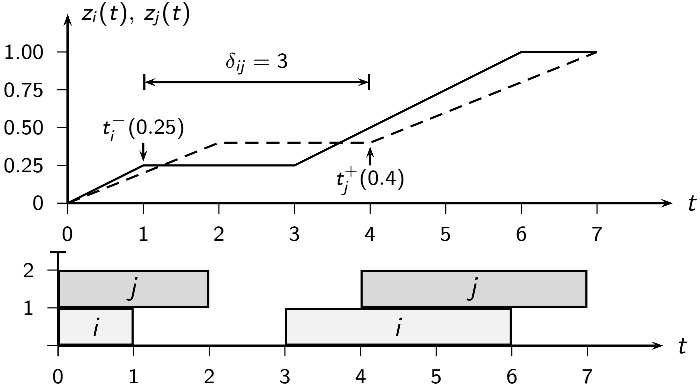
   $$(i,j,\xi_i,\xi_j) : \max\{t \mid z_j(t) = \xi_j\} \geq \min\{t \mid z_i(t) = \xi_i\}$$

5. Generalized feeding precedence relations (S. and Paetz 2014)

   $$(i,j,\xi_i,\xi_j,\delta_{ij}) : \underbrace{\max\{t \mid z_j(t) = \xi_j\}}_{t_j^+(\xi_j)} \geq \underbrace{\min\{t \mid z_i(t) = \xi_i\}}_{t_i^-(\xi_i)} + \delta_{ij}$$

Example

Generalized feeding precedence relation $(i, j, 0.25, 0.4, 3)$



Resource constraints

- Different types of resources considered in project scheduling: renewable, nonrenewable, doubly-constrained, storage, partially renewable, continuous resources
- In this talk: renewable resources k from a set $\mathcal{R}$
  - Each resource $k \in \mathcal{R}$ consists of $R_k \in \mathbb{N}$ identical units (capacity)
  - Each activity uses $r_{ik} \in \mathbb{Z}_{\geq 0}$ units when being in progress
- Resource constraints: joint requirements of activities i for resources must not exceed the resource capacities at any point in time

$$\sum_{i \in V} r_{ik} y_i(t) \leq R_k \quad (k \in \mathcal{R}; t \geq 0)$$

Objective functions

- Scheduling goals specified by single or several objective functions f
- In this talk: single-criterion problems
- Regular objective function

$$C \leq C' \Rightarrow f(C) \leq f(C')$$

  - Project duration $f(C) = \max_{i \in V} C_i$
  - Total tardiness cost $f(C) = \sum_{i \in V} w_i (C_i - d_i)^+$
- Nonregular objective functions
  - Net present value $f(C) = \sum_{i \in V} c_i^F \beta^{C_i}$
  - Total squared utilization cost (resource leveling) $f(y) = \sum_{k \in \mathcal{R}} c_k \int_0^\infty \left(\sum_{i \in V} r_{ik} y_i(t) \right)^2 dt$

Resource-constrained project scheduling problem

- General project scheduling problem with generalized (feeding) precedence relations and renewable-resource constraints

$$(\overline{P}) \begin{cases} \text{Min.} & f(y) \\ \text{s. t.} & \int_0^\infty y_i(t)\,dt = p_i \qquad (i \in V) \\ & t_j^+(\xi_j) \geq t_j^-(\xi_i) + \delta_{ij} \qquad ((i,j) \in E) \\ & \sum_{i \in V} r_{ik} y_i(t) \leq R_k \qquad (k \in \mathcal{R}; t \geq 0) \end{cases}$$

- Non-preemptive version with $\mathcal{A}(S,t) := \{i \in V \mid S_i \leq t < S_i + p_i\}$

$$(P) \begin{cases} \text{Min.} & f(S) \\ \text{s. t.} & S_j \geq S_i + \delta_{ij} \qquad ((i,j) \in E) \\ & \sum_{i \in \mathcal{A}(S,t)} r_{ik} \leq R_k \qquad (k \in \mathcal{R}; t \geq 0) \\ & S_i \geq 0 \qquad (i \in V) \end{cases}$$

Set of feasible schedules: $\mathcal{S}$, set of time-feasible schedules: $\mathcal{S}_T$

Time-constrained project scheduling problem

$$(P_T) \begin{cases} \text{Min.} & f(S) \\ \text{s. t.} & S_j \geq S_i + \delta_{ij} \quad ((i,j) \in E) \\ & S_i \geq 0 \quad (i \in V) \end{cases}$$

Generalized precedence relations (i, j, δ_{ij}) represent minimum and maximum time lags between starts of activities i and j

- $\delta_{ij} = p_i$: ordinary precedence relation $S_j \geq S_i + p_i = C_i$
- $\delta_{ij} > p_i$: delayed precedence relation $S_j \geq C_i + (\delta_{ij} - p_i)$
- $0 \leq \delta_{ij} < p_i$: minimum time lag allowing overlapping of i and j
- $\delta_{ij} < 0$: maximum time lag of $-\delta_{ij}$ between starts of j and i

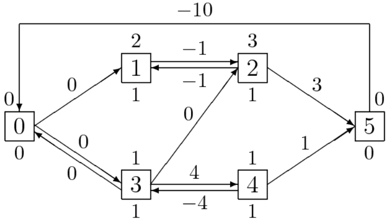
$$S_j \geq S_i + \delta_{ij} \Leftrightarrow S_i \leq S_j - \delta_{ij}$$

Completion-to-start, completion-to-completion, and start-to-completion time lags can be transformed into start-to-start time lags

MPM (activity-on-node) project network (Roy 1964)

- Generalized precedence relations represented by MPM network $N = (V, E, \delta)$
- Activities correspond to nodes, precedence relations to arcs
- Introduce nodes 0 and $n+1$ for project beginning and termination
- Nonnegativity conditions $S_i \geq 0$ can be replaced by $S_0 = 0$

Example: MPM network for project with four real activities



Modeling practical constraints

- Release date r_i of activity i: $\delta_{0i} = r_i$
- Quarantine time q_i of activity i: $\delta_{i(n+1)} = p_i + q_i$
- Deadline $\overline{d}_i$ for completion of activity i: $\delta_{i0} = -\overline{d}_i + p_i$
- Fixed start time t_i for activity i: $\delta_{0i} = t_i, \delta_{i0} = -t_i$
- Simultaneous start of activities i and j: $\delta_{ij} = \delta_{ji} = 0$
- Simultaneous completion of activities i and j: $\delta_{ij} = p_i - p_j, \delta_{ji} = p_j - p_i$
- Processing activities i, j immediately one after another: $\delta_{ij} = p_i, \delta_{ji} = -p_i$
- Minimum overlapping time ℓ_{ij} of i and j: $\delta_{ij} = \ell_{ij} - p_i$, $\delta_{ji} = \ell_{ij} - p_j$
- Maximum makespan C^U_{max} for activity set U: $\delta_{ij} = -C^U_{max} + p_i$ for $i, j \in U$
- Time-varying resource capacities: dummy activities with fixed start times
- Time-varying resource requirements: sequence of sub-activities pulled tight
- ...

Temporal analysis with MPM

- MPM: Metra Potential Method
- Interpret project network as electric circuit
- Potential: assignment $S : V \to \mathbb{R}_{\geq 0}$
- Tensions: differences $S_j - S_i$ of potentials
- Generalized precedence relations: lower bounds δ_{ij} on tensions $S_j - S_i$
- Dual (D) of problem (P_T) with $f(S) = \sum_{i \in V \setminus \{0\}} S_i - (n+1) S_0$

$$(D) \begin{cases} \text{Max.} & \sum_{(i,j)\in E} \delta_{ij} \cdot \varphi_{ij} \\ \text{s. t.} & \sum_{(i,j)\in E} \varphi_{ij} - \sum_{(j,i)\in E} \varphi_{ji} = \begin{cases} -1 \text{ for } i \in V \setminus \{0\} \\ (n+1) \text{ for } i = 0 \end{cases} \\ & \varphi_{ij} \geq 0 \quad ((i,j) \in E) \end{cases}$$

is longest-walk problem in N

Temporal analysis with MPM

Fundamentals

- $\mathcal{S}_T \neq \emptyset$ iff N does not contain any cycle of positive length
- Induced time lag $d_{ij} := \min_{S \in \mathcal{S}_T}(S_j - S_i) =$ length of longest walk from i to j in N ("distance")
- Earliest start time $ES_i = d_{0i}$, latest start time $LS_i = -d_{i0}$

Algorithms and complexities (with $m := |E|$)

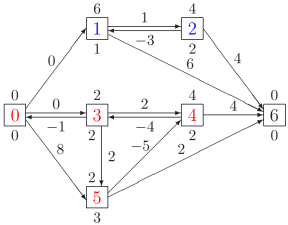
- $\mathcal{S}_T \neq \emptyset$ and single time lag d_{ij}: transformation of Bianco and Caramia (2010) to unit-capacity transshipment problem, $\mathcal{O}(m)$
- All time lags (distance matrix $D = (d_{ij})_{i,j \in V}$): Floyd-Warshall-Algorithm, $\mathcal{O}(n^3)$
- Update of distance matrix after increase of single d_{ij}: Algorithm of Bartusch et al. (1988), $\mathcal{O}(n^2)$
- Earliest and latest schedules ES and LS: label-correcting algorithm for longest-walk calculations, $\mathcal{O}(mn)$

Resource-constrained problem (P): Complexity and decomposition

- Problem (P) is $\mathcal{NP}$-hard
- The feasibility variant of problem (P) is $\mathcal{NP}$-complete

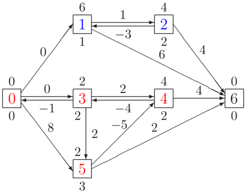
Decomposition theorem (Neumann and Zhan 1995)

An instance of problem (P) is feasible if and only if for each strong component G of project network N there exists a feasible subschedule for the execution of all activities of G.

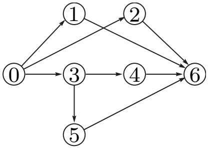


- Classical schedule-generation schemes must be modified to avoid or to cope with deadlocks
- Decomposition theorem is basis of heuristic decomposition methods

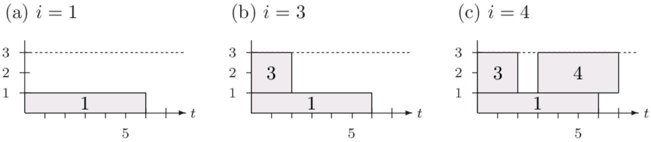
Example



Precedence graph



Assume $R = 3$ and schedule activities with serial schedule-generation scheme



- Deadlock for activity $i = 2$ after three iterations
- Conclusion: start times of activities cannot be fixed during scheduling

Bartusch's Lemma

- Forbidden set $F \subseteq V$: $\sum_{i \in F} r_{ik} > R_k$ for some $k \in \mathcal{R}$
- Forbidden set F broken up by schedule S: $\mathcal{A}(S,t) \not\supseteq F$ for all $t \geq 0$

Lemma (Bartusch et al. 1988)

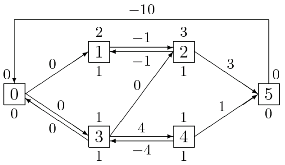
1. An $\subseteq$-minimal forbidden set F is broken up by schedule S iff F contains two activities i, j with $S_j \geq S_i + p_i$.
2. Schedule S is resource-feasible iff all $\subseteq$-minimal forbidden sets F are broken up.

Consequences:

- Resource constraints can be expressed as disjunctions of ordinary precedence relations (i, j)
- Feasible region is union of finitely many relation polyhedra

$$\mathcal{S}_T(\rho) = \{S \in \mathcal{S}_T \mid S_j \geq S_i + p_i \text{ for all } (i,j) \in \rho\}$$

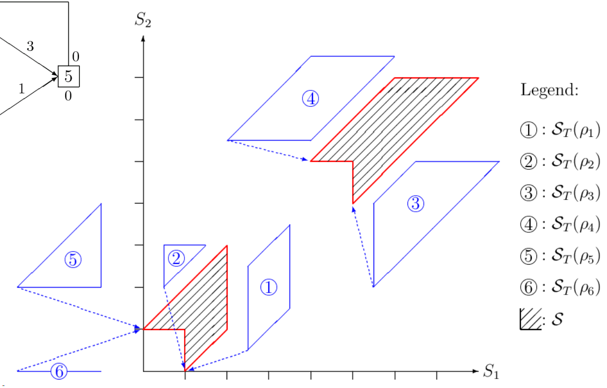
Covering of $\mathcal{S}$ by relation polyhedra



Capacity $R = 2$

$F_1 = \{1, 2, 3\}$,
$F_2 = \{1, 2, 4\}$

1) $3 \to 1, 1 \to 4$
2) $3 \to 1, 2 \to 4$
3) $3 \to 1, 4 \to 1$
4) $3 \to 1, 4 \to 2$
5) $3 \to 2, 1 \to 4$
6) $3 \to 2, 2 \to 4$



Feasible relations (S. 2005)

Definition: Feasible relation

Relation ρ with $\emptyset \neq \mathcal{S}_T(\rho) \subseteq \mathcal{S}$ is called feasible relation.

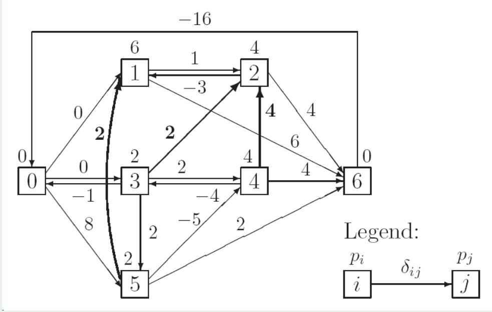
- Condition $\mathcal{S}_T(\rho) \neq \emptyset$: ordinary precedence relations $(i, j) \in \rho$ are compatible with generalized precedence relations $(i', j') \in E$
- Condition $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$: all schedules S satisfying the ordinary precedence relations $(i, j) \in \rho$ are resource-feasible

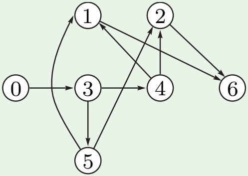
Induced strict order, schedule-induced order, iso-order set

- Relation network $N(\rho) = (V, E \cup \rho, \delta)$ with $\delta_{ij} = p_i$ for $(i, j) \in \rho$
- Distance matrix $D(\rho)$ associated with network $N(\rho)$
- Relation ρ induces strict order $\Theta(\rho) := \{(i, j) \mid d_{ij}(\rho) \geq p_i\}$
- Schedule S induces strict order $\theta(S) := \{(i, j) \mid S_j \geq S_i + p_i\}$
- Iso-order set $\mathcal{S}_T^{=}(\theta) := \{S \in \mathcal{S}_T \mid \theta(S) = \theta\}$

Example

Relation network for $\rho = \{(3,2),(4,2),(5,1)\}$ and strict order $\Theta(\rho)$

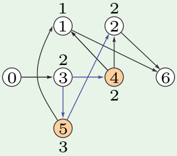


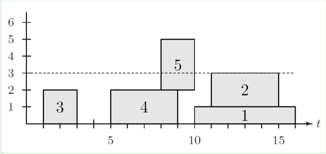


Checking feasibility of relations (Kaerkes and Leipholz 1977)

1. $\mathcal{S}_T \neq \emptyset$ iff $N(\rho)$ does not contain any cycle of positive length
2. $\mathcal{S}_T(\rho) \subseteq \mathcal{S}$:
   - For each resource k weight activities $i \in V$ with r_{ik}
   - Condition is satisfied iff for each resource k, weight of any antichain $\mathcal{A}_k(\rho)$ of $\Theta(\rho)$ does not exceed R_k
   - Maximum-weight antichain can be computed in $\mathcal{O}(n^3)$ time by solving maximum-cut problem in precedence graph of $\Theta(\rho)$

Example: Maximum-weight antichain for $\rho = \{(3,2),(4,2),(5,1)\}$

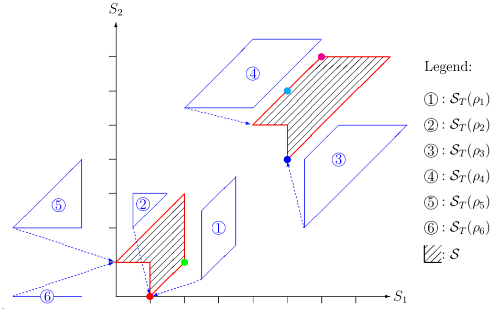




- Weight nodes with requirements r_{ik}
- Determine maximum $0-(n+1)$-cut
- Here: $\mathcal{A}(\rho) = \{4,5\}$

Schedule types (Neumann et al. 2000)

- Active schedules: Minimal points of $\mathcal{S}$
- Stable schedules: Extreme points of $\mathcal{S}$
- Pseudostable schedules: Local extreme points of $\mathcal{S}$
- Quasiactive schedules: Minimal points of relation polyhedra
- Quasistable schedules: Vertices of relation polyhedra



Classes of objective functions and efficient solutions

- Regular functions $\rightarrow$ project duration S_{n+1}
- Linear(-izable) functions $f(S)$
  $\rightarrow$ subset makespan $\max_{i \in U}(S_i + p_i) - \min_{i \in U} S_i$
- Binary monotonic functions: monotonicity in binary directions $s \in \{0,1\}^n$
  $\rightarrow$ net present value $\sum_{i \in V} c_i^F \beta^{S_i + p_i}$
- Locally regular functions: f regular on iso-order sets $\mathcal{S}_T^{=}$
  $\rightarrow$ total resource availability cost $\sum_{k \in \mathcal{R}} c_k \max_{t \geq 0} r_k(S,t)$
- Locally concave functions: f concave on iso-order sets $\mathcal{S}_T^{=}$
  $\rightarrow$ total squared utilization cost $\sum_{k \in \mathcal{R}} c_k \int_0^{\infty} r_k^2(S,t)\, dt$

Objective function	Efficient solutions	Verification
Regular	Active schedules	$\mathcal{NP}$-complete
Linear	Stable schedules	$\mathcal{NP}$-complete
Binary monotonic	Pseudostable schedules	$\mathcal{NP}$-complete
Locally regular	Quasiactive schedules	polynomial
Locally concave	Quasistable schedules	polynomial

Generic solution approaches

Regular, linear, and binary-monotonic objective functions

- Time-constrained scheduling problem (P_T) efficiently solvable
  - longest-walk calculations
  - linear programming
  - recursive algorithms, e. g., De Reyck and Herroelen (1998b)
  - steepest-descent algorithms, e. g., S. and Zimmermann (2001)
- Apply relaxation-based procedure providing feasible relation ρ
- Minimize f on relation polyhedron $\mathcal{S}_T(\rho)$

Objective function (only) locally regular or locally concave

- Time-constrained scheduling problem (P_T) intractable
- Apply schedule-construction procedure providing minimal or extreme point of some relation polyhedron $\mathcal{S}_T(\rho)$

Relaxation-based procedure

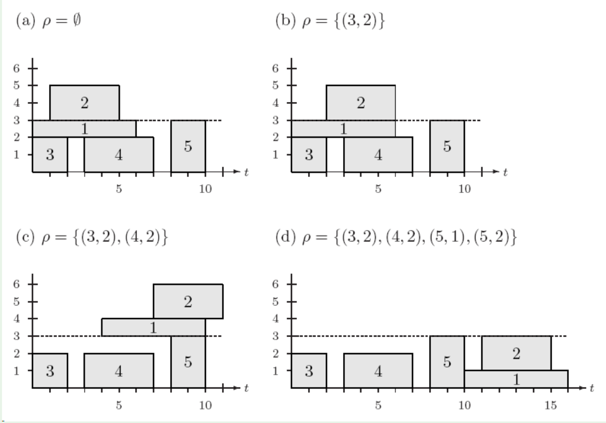
Schedule-generation scheme for problem (P)

1. Set $\rho := \emptyset$;
2. If $\mathcal{S}_T(\rho) = \emptyset$: STOP; // no feasible schedule found
3. Verify feasibility of ρ by solving maximum-cut problems;
4. If ρ is feasible: compute minimizer of f on $\mathcal{S}_T(\rho)$ and STOP;
5. Determine resource k such that antichain $\mathcal{A}_k(\rho)$ is forbidden;
6. Select $\subseteq$-minimal set $B \subset \mathcal{A}_k(\rho)$ such that $A := \mathcal{A}_k(\rho) \setminus B$ is not forbidden, and select some $i \in A$;
7. Set $\rho := \rho \cup (\{i\} \times B)$, and go to step 2;

- Combination (i, B) is called a minimal delaying mode (De Reyck and Herroelen 1998a)
- Procedure can also be used for problems with stochastic processing times $\widetilde{p}_i$; resulting relation defines an *ES*-policy (Radermacher 1981)

Example

Relaxation-based procedure



Schedule-construction procedure

Schedule-generation scheme for problem (P_T)

1. Set $\mathcal{C} := \{0\}$, $S_0 := 0$, and $ES_i := d_{0i}$, $LS_i := -d_{i0}$ for all $i \in V$;
2. Select some $i \in \mathcal{C}$ and some $j \in V \setminus \mathcal{C}$;
3. Select time $S_j \in \{S_i + \delta_{ij}, S_i + p_i, S_i - p_j, S_i - \delta_{ji}\} \cap [ES_j, LS_j]$;
4. Add j to $\mathcal{C}$;
5. Update ES_h and LS_h for all $h \in V \setminus \mathcal{C}$;
6. If $\mathcal{C} \neq V$, go to step 2;

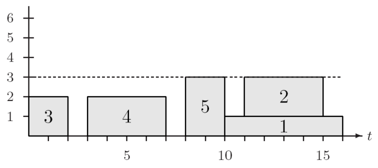
- Locally regular objective function: select $t \in \{S_i + \delta_{ij}, S_i + p_i\}$
- Pairs (i, j) selected in step 2 form a spanning tree (spanning outtree) of relation network $N(\rho)$ rooted at node 0
- In case of resource constraints: determine $\subseteq$-minimal feasible relation ρ and apply procedure on network $N(\rho)$ instead of N

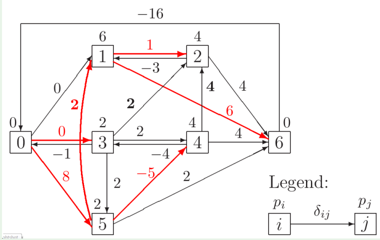
Example

Schedule-construction procedure for quasiactive schedule

Iteration	i	j	time t
1	0	3	0
2	0	5	8
3	5	4	3

Iteration	i	j	time t
4	5	1	10
5	1	2	11
6	1	6	16





Solution approaches for the project duration problem

- Minimization of project duration has received largest attention in literature
- Four categories of solutions approaches
  - Adaptations of schedule-construction procedures and metaheuristics for RCPSP
  - Relaxation-based branch-and-bound procedures
  - Constraint-programming based approaches
  - Mixed-integer linear programming formulations and related algorithms

Schedule-construction procedures

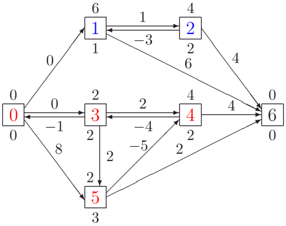
- Serial/parallel SGS iteratively fix start times of activities
- When procedure is trapped in deadlock: call unscheduling procedure
- No guarantee to find a feasible solution, but very effective on benchmark instances

Unscheduling procedure (Franck et al. 2001)

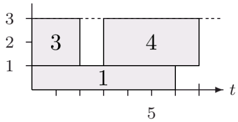
1. Set $\Delta := t - LS_j$; // t is earliest resource-feasible start // time of j
2. Determine $\mathcal{U} := \{i \in \mathcal{C} \mid LS_j = S_i - d_{ji}\}$;
3. For all $i \in \mathcal{U}$: set $ES_i := ES_i + \Delta$;
4. Remove all h with $S_h \geq \min_{i \in \mathcal{U}} S_i$ from set $\mathcal{C}$;
5. Update earliest and latest start times and return to schedule-generation scheme;

- Priority-rule based methods, tabu search, and genetic algorithm by Franck et al. (2001) and evolutionary algorithm by Ballestín et al. (2011) based on serial SGS with unscheduling

Example

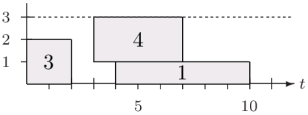


(c) $i = 4$

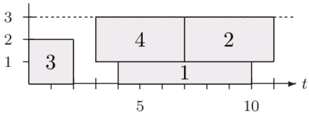


- $j = 2,\ t = 7,\ LS_j = 3,\ \Delta = 4$
- $\mathcal{U} = \{i \in \mathcal{C} \mid LS_j = S_i - d_{ji}\} = \{1\}$
- Set $ES_1 := 4$ and unschedule activities $i = 1, 3, 4$

(a) $i = 3, 4, 1$



(b) $i = 2$



Relaxation-based approaches

- Start with solution $\widehat{S} = ES$ to time-constrained problem (P_T)
- Identify some time t with $r_k(\widehat{S}, t) > R_k$ for some resource k
- Branch over alternatives to resolve the resource conflict at time t
- Partition forbidden set $\mathcal{A}(\widehat{S}, t)$ in minimal delaying alternative B and feasible set $A = \mathcal{A}(\widehat{S}, t) \setminus B$

  - Ordinary precedence relations (De Reyck and Herroelen 1998a)

    $$S_j \geq S_i + p_i \quad (j \in B) \qquad \text{for some } i \in A$$

  - Release dates (Fest et al. 1999)

    $$S_j \geq \delta_{0j} := \min_{i \in A}(\widehat{S}_i + p_i) \quad (j \in B)$$

  - Disjunctive precedence relations (S. 1998)

    $$S_j \geq \min_{i \in A}(S_i + p_i) \quad (j \in B)$$

Constraint-programming approaches
(Dorndorf et al. 2000, Schutt et al. 2013)

- Associate decision variables S_i with domains $\Delta_i = \{ES_i, \ldots, LS_i\}$
- Try to reduce domain sizes by applying consistency tests like precedence, interval capacity, or disjunctive consistency tests
- When consistency tests reach fixed point, perform dichotomic start-time branching for activity i with smallest earliest start time $t = \min \Delta_i$: $S_i = t \vee S_i \geq t + 1$
- Replace domain Δ_i by $\{t\}$ or $\{t + 1, \ldots, LS_i\}$
- Propagate update to other domains by applying consistency tests

Best results for project duration problem obtained by Schutt et al. (2013) from combining start-time branching with SAT representation and lazy clause generation

Alternative approach by Cesta et al. (2002) based on formulation as CSP for posting precedence relations in minimal forbidden sets

Mixed-integer linear programming (Bianco and Caramia 2012)

- In general, MILP formulation for RCPSP easily adapted to generalized precedence relations
- MILP model of Bianco and Caramia (2012)
  - Binary variables $s_{it} = 1$ if i has been started by time t
  - Binary variables $f_{it} = 1$ if i has been completed by time t
  - Variables $z_{it} \in [0, 1]$ keeping execution percentage of i by time t
  - Coupling constraints: $z_{i(t+1)} - z_{it} = \frac{1}{p_i}(s_{it} - f_{it})$
  - Temporal constraints: $\sum_{t=1}^{T} s_{it} \geq \sum_{t=1}^{T} s_{jt} + \delta_{ij}$
  - Resource constraints: $\sum_{i \in V} r_{ik} p_i \cdot (z_{it} - z_{i(t-1)}) \leq R_k$
- Branch-and-bound algorithm based on MILP formulation
  - Each level of enumeration tree associated with one activity i
  - Branch over $\vee_{t=ES_i,\ldots,LS_i}\{s_{it} = 1\}$
  - Lower bounds obtained by Lagrangian relaxation of resource constraints

Experimental performance analysis for project duration problem

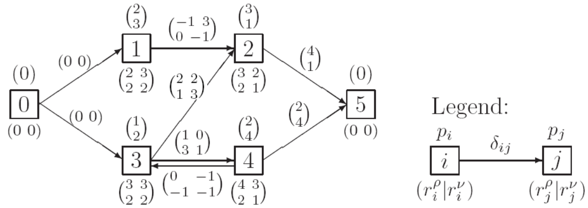
- Algorithms evaluated on ProGen/max data sets[1]
- Results for test set CD (540 instances, $n = 100$, $|\mathcal{R}| = 5$)

Algorithm	t_{cpu}	p_{feas}	p_{opt}	p_{inf}
De Reyck and Herroelen (1998a)	3	97.3	54.8	1.4
	30	97.5	56.4	1.4
S. (1998)	3	98.1	58.0	1.9
	30	98.1	62.5	1.9
	100	98.1	63.4	1.9
Fest et al. (1999)	3	92.2	58.1	1.9
	30	98.1	69.4	1.9
	100	98.1	71.1	1.9
Dorndorf et al. (2000)	3	97.8	66.2	1.9
	30	98.1	70.4	1.9
	100	98.1	71.1	1.9
Bianco and Caramia (2012)	3	98.1	67.6	1.9
	30	98.1	71.8	1.9
	100	98.1	72.2	1.9
Schutt et al. (2013)	1	97.9	78.1	1.6
	10	98.1	89.8	1.9
	100	98.1	94.0	1.9

[1] http://www.wiwi.tu-clausthal.de/en/abteilungen/produktion/forschung/schwerpunkte/project-generator/

The multi-mode version of problem (P)

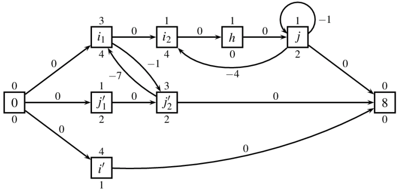
- Each activity i can be executed in one of a finite number of execution modes $m \in \mathcal{M}_i$
- Executions modes m differ in durations p_{im} and resource requirements r_{ikm} (renewable and nonrenewable resources)
- Generalized precedence relations δ_{ij} depend on modes m_i and m_j

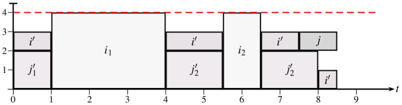


- Feasibility variant of time-constrained problem (P_T) $\mathcal{NP}$-complete
- Relaxation-based branch-and-bound algorithms by De Reyck and Herroelen (1999) and Heilmann (2003), MILP model by Sabzehparvar and Seyed-Hosseini (2008)

Preemptive problem ($\overline{P}$) (S. and Paetz 2014)

- Activities can be interrupted at any point in time
- Generalization of problem (P) since preemption can be prevented by generalized feeding precedence relations of type $(i, i, 1.0, 0.0, -p_i)$
- $(\overline{P})$ can be reduced to canonical form with nonpositive completion-to-start time lags
- Up to $2n - 1$ slices needed, one and the same antichain can be in progress several times, number of interruptions bounded by $n(n-1)$
- Subproblem with given positive antichains still $\mathcal{NP}$-hard





Practical applications including generalized precedence relations

- Technical constraints in civil engineering (Bartusch et al. 1988)
- Lot streaming in manufacturing (Neumann and S. 1997)
- Perishable intermediate products in process scheduling (Neumann et al. 2002)
- Minimum and maximum durations of service activities (Mellentien et al. 2004)
- Minimum and maximum time lags between build-up and test activities in automotive R&D projects (Bartels and Zimmermann 2009)
- Overlapping of activities in aggregate production scheduling (Alfieri et al. 2011)
- Maximum duration of validity for statutory permissions in nuclear power plant dismantling (Bartels et al. 2011)
- Maximum makespan for activity sequences at service centers (Quintanilla et al. 2012)

Conclusions

- Generalized precedence relations needed to formulate real-life scheduling constraints
- Efficient temporal analysis based on Roy's Metra Potential Method
- Feasibility variant of resource-constrained problems $\mathcal{NP}$-complete
- Classical schedule-generation schemes lead to deadlocks
- Unscheduling techniques, relaxation-based approaches, constraint programming methods, mixed-integer programming formulations
- Significant recent advances, e. g.:
  - Linear-time algorithm for checking feasibility of temporal constraints
  - Very effective constraint-programming approaches for project duration problem
- Avenues for future research
  - Preemptive project scheduling under gpr's
  - Stochastic/robust project scheduling under gpr's
  - Lazy clause generation approach for different objective functions

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